be $(h_1, h_2, h_3, h_a) = (0, 0, \mp h, 0)$, and the desired final state would be $(h_1, h_2, h_3, h_a) = (-h, 0, 0, h)$.

Significance of the Transformation

Unifying transformations of this kind are always of academic interest and often of practical importance as well. This result is interesting because it helps us to understand the similarities and differences between the different spinup problems, but it is also very useful. An immediate application is in developing a graphics program to generate three-dimensional views of the momentum sphere. Since the sphere associated with a prolate spacecraft may be obtained by inverting and rotating the sphere for an equivalent oblate spacecraft, it is unnecessary to deal specifically with the prolate case, resulting in a simpler program design.

Another application is in the development of a perturbation analysis of Eqs. (1-4) for small g_a. Since the $g_a = 0$ solution involves Jacobi's elliptic functions,[8] a perturbation treatment can be quite complicated.[6] The oblate/prolate equivalence reduces the number of cases that must be considered.

Conclusions

The distinction between oblate and prolate dual spinners has been made by most researchers of these problems. In this Note we presented a simple transformation that relates the global dynamics of the two types of spacecraft. We also pointed out the significance of the result and offered some sample applications.

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Stability of a Rate Gyro

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Nomenclature

A, C = transverse and axial moments of inertia of rotor

A_g, B_g, C_g = principal moments of inertia of gimbal

C_d, K = damping coefficient and spring constant, respectively

$C_d\dot{\theta}, K\theta$ = terms of first degree of Taylor series expansions of $f_1(\dot{\theta})$ and $f_2(\theta)$, respectively

D_1, D_2, D_3, D_4 = $C_d/n(A + A_g)$, $K/n^2(A + A_g)$, $C/n(A + A_g)$, $(A + B_g - C_g)/n^2(A + A_g)$, respectively

$f_1(\dot{\theta}), f_2(\theta)$ = nonlinear torsional damping torque and spring torque, respectively

$m_1, m_2, \lambda_1, \lambda_2, \lambda_3$ = undetermined positive constants

n = rated angular velocity of rotor, $\dot{\psi} - \omega_Y \sin\theta + \omega_Z \cos\theta$

OX, OY, OZ = orthogonal axes fixed with platform, where OY is input axis, OX is output axis of gimbal and always coincides with Ox

Ox, Oy, Oz = principal axes of inertia of rotor and gimbal, which coincide with OX, OY, OZ, respectively, in the equilibrium position

α, β = θ, θ', respectively

θ = output deflection angle between rotor axis Oz and OZ

θ', θ'' = $d\theta/d\tau$, $d^2\theta/d\tau^2$, respectively

τ = dimensionless time, nt

$\dot{\psi}$ = spin angular velocity of rotor

$\omega_X, \omega_Y, \omega_Z$ = components of angular velocity of rotor along OX, OY, OZ, respectively, where ω_Y is the input

Introduction

FOR many practical applications, it is necessary to measure the angular velocity of a given vehicle. The angular velocity about an axis normal to a given platform can be recorded by means of a single-gimbal gyro whose spin axis is normal to the axis of motion. Such a gyro is referred to as a rate gyro.

Using Lagrange's equation, we can derive the following equation of motion for the output deflection angle θ of a rate gyro[1] (see Fig. 1):

$$(A + A_g)\ddot{\theta} + f_1(\dot{\theta}) + f_2(\theta) + Cn(\omega_Y \cos\theta + \omega_Z \sin\theta) + (A + B_g - C_g)(\omega_Y \cos\theta + \omega_Z \sin\theta)(\omega_Y \sin\theta - \omega_Z \cos\theta) = -(A + A_g)\dot{\omega}_X \tag{1}$$

where

$$Cn = C(\dot{\psi} - \omega_Y \sin\theta + \omega_Z \cos\theta) = \text{const} \tag{2}$$

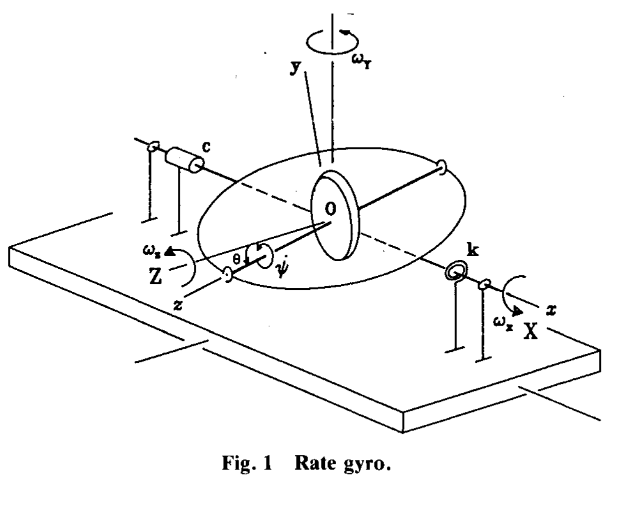

Fig. 1 Rate gyro.

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Since we are interested in the stability behavior, it is assumed that the angular velocity ω_Y about the input axis OY is zero and the platform undergoes a steady rotation about the OX axis, i.e., $\dot{\omega}_X = 0$.[1] Equation (1) now can be simplified to

$$(A + A_g)\ddot{\theta} + C_d\dot{\theta} + K\theta + Cn\omega_Z \sin\theta - (A + B_g - C_g)\omega_Z^2 \sin\theta\cos\theta + \cdots = 0 \tag{3}$$

The terms of the series expansion higher than first order are included in the ellipsis of Eq. (3). For convenience, we change Eq. (3) into the dimensionless form:

$$\ddot{\theta} + D_1\theta' + D_2\theta + D_3\omega_Z(\tau)\sin\theta - D_4\omega_Z^2(\tau)\sin\theta\cos\theta + \cdots = 0 \tag{4}$$

If we set $\alpha = \theta$ and $\beta = \theta'$, then Eq. (4) becomes the following nonlinear, nonautonomous, ordinary differential equations:

$$\alpha' = \beta$$

$$\beta' = -\left[D_2\alpha + D_3\omega_Z(\tau)\sin\alpha - D_4\omega_Z^2(\tau)\sin\alpha\cos\alpha\right] - D_1\beta + \cdots \tag{5}$$

Since the stability of the relative equilibrium $\alpha = \beta = 0$ is studied, Eq. (5) is also the differential equations of disturbances.

Since Liapunov functions applicable to stability problems are commonly shown to be quadratic forms,[2,3] we try the Liapunov function:

$$V = \frac{\lambda_1}{2}\alpha^2 + \lambda_2\alpha\beta + \frac{\beta}{2} \tag{6}$$

where λ_1 and λ_2 are undetermined positive constants. Then we have the negative time derivative of V through Eq. (5):

$$-\frac{dV}{d\tau} = L = W^* \tag{7}$$

where

$$L = \lambda_2\left[D_2 + D_3\omega_Z(\tau) - D_4\omega_Z^2(\tau)\right]\alpha^2 + \left[\lambda_2 D_1 + D_2 + D_3\omega_Z(\tau) - \lambda_1 - D_4\omega_Z^2(\tau)\right]\alpha\beta + (D_1 - \lambda_2)\beta^2 \tag{8}$$

and W^* represents the terms of higher degree.

Since L contains time τ explicitly, we must find a function W that does not contain τ explicitly such that $L \geq W$.[2,3] We try

$$W = (D_2 + D_3\lambda_3)\lambda_2\alpha^2 + (D_2 + D_3\lambda_3 + D_1\lambda_2 - \lambda_1)\alpha\beta + \lambda_2\beta^2 \tag{9}$$

where λ_3 is an undetermined constant that has the same dimension as $\omega_Z(\tau)$. By Sylvester's theorem,[2-4] we know that W is positive definite if

$$\lambda_3 > -\frac{D_2}{D_3} = -\frac{K}{Cn} \tag{10a}$$

$$D_1\lambda_2 + D_2 + D_3\lambda_3 - 2\lambda_2\sqrt{D_2 + D_3\lambda_3} < \lambda_1 < D_1\lambda_2 + D_2 + D_3\lambda_3 + 2\lambda_2\sqrt{D_2 + D_3\lambda_3} \tag{10b}$$

Furthermore,

$$L - W = \lambda_2\left\{D_3\left[\omega_Z(\tau) - \lambda_3\right] - D_4\omega_Z^2(\tau)\right\}\alpha^2 + \left\{D_3\left[\omega_Z(\tau) - \lambda_3\right] - D_4\omega_Z^2(\tau)\right\}\alpha\beta + (D_2 - 2\lambda_2)\beta^2$$

If the following inequalities hold, then $L - W \geq 0$ ($=0$ only when $\alpha = \beta = 0$), i.e., L is positive definite, and $L - W^*$ is also positive definite[3]:

$$\frac{D_3 - \sqrt{D_3^2 - 4D_3D_4\lambda_3}}{2D_4} < \omega_Z < \frac{D_3 + \sqrt{D_3^2 - 4D_3D_4\lambda_3}}{2D_4} \tag{11a}$$

$$\frac{D_3}{4D_4} \geq \lambda \tag{11b}$$

$$D_1 > 2\lambda_2 \tag{11c}$$

$$\lambda_3 < \frac{D_3}{4D_4} - \frac{4\lambda_2(D_1 - 2\lambda_2)}{D_3} \tag{11d}$$

$$\omega_Z(\tau) < \frac{D_3 - \sqrt{D_3^2 - 4D_4\left[D_3\lambda_3 + 4\lambda_2(D_1 - 2\lambda_2)\right]}}{2D_4}$$

or

$$\omega_Z(\tau) > \frac{D_3 + \sqrt{D_3^2 - 4D_4\left[D_3\lambda_3 + 4\lambda_2(D_1 - 2\lambda_2)\right]}}{2D_4} \tag{11e}$$

By Sylvester's theorem, the sufficient condition for the positive definiteness of V is

$$\lambda_1 > \lambda_2^2 \tag{12}$$

By Eqs. (10b), (11c), and (12), we have

$$\lambda_2^2 < \lambda_1 < 2\lambda_2^2 + D_2 + D_3\lambda_3 + 2\lambda_2\sqrt{D_2 + D_3\lambda_3} < D_1\lambda_2 + D_2 + D_3\lambda_3 + 2\lambda_2\sqrt{D_2 + D_3\lambda_3} \tag{13}$$

so λ_1 can be chosen. Also, from Eqs. (10a), (11b), and (11d), we know that λ_2 can be chosen because

$$-\frac{D_1}{D_3} < \lambda_3 < \frac{D_3}{4D_4} - \frac{4\lambda_2(D_1 - 2\lambda_2)}{D_3} \tag{14}$$

holds if we properly choose λ_2.

From Eqs. (11a) and (11e), by choosing λ_2 and λ_3 properly, we get the conditions that assure that V and $dV/d\tau$ are positive definite:

$$\frac{D_3 - \sqrt{D_3^2 + 4D_2D_4}}{2D_4} < \omega_Z < \frac{D_3}{2D_4}$$

or

$$\frac{D_3}{2D_4} < w_Z < \frac{D_3 + \sqrt{D_3^2 + 4D_2D_4}}{2D_4} \tag{15}$$

Now, since V does not contain τ explicity, it permits an infinitesimal upper limit. When Eq. (15) is satisfied by Liapunov's theorem of asymptotic stability for nonautonomous system,[2-4] the relative equilibrium $\alpha = \beta = 0$ is asymptotically stable.

Since Liapunov functions applicable to instability problems are commonly chosen as the product of variables,[2,3] we try the Liapunov function:

$$V = \alpha^{m_1}\beta^{m_2} \tag{16}$$

where m_1 and m_2 are undetermined positive constants. Then we have

$$\frac{dV}{d\tau} = \alpha^{m_1 - 1}\beta^{m_2 - 1}(L + W^*) \tag{17}$$

where

$$L = \left[D_4m_2\omega_Z^2(\tau) - D_3m_2\omega_Z(\tau) - D_2m_2\right]\alpha^2 - D_1m_2\alpha\beta + m_1\beta^2 \tag{18}$$

Table 1 Liapunov function V used to prove asymptotic stability and instability

V	Differential equations of disturbances	Sufficient conditions for asymptotic stability (a.s.) or instability (ins.) of $\alpha=\beta=0$
$\frac{\lambda_1}{2}\alpha^2+\lambda_2\alpha\beta+\frac{\beta}{2}$	(5)	$\frac{D_3-\sqrt{D_3^2+4D_2D_4}}{2D_4}<\omega_Z<\frac{D_3}{2D_4}$ or
		$\frac{D_3}{2D_4}<\omega_Z<\frac{D_3+\sqrt{D_3^2+4D_2D_4}}{2D_4}$ (a.s.)
$\alpha^{m_1}\beta^{m_2}$	(5)	$\omega_Z(\tau)<\frac{D_3-\sqrt{D_3^2+4D_3D_4}}{2D_4}$ or
		$\frac{D_3+\sqrt{D_3^2+4D_3D_4}}{2D_4}<\omega_Z(\tau)$ (ins.)
$\frac{\lambda_1}{2}\alpha^2+\lambda_2\alpha\beta+\frac{\beta}{2}$	(25)	$\omega_Z>-K/Cn$ (a.s.)
$\alpha^{m_1}\beta^{m_2}$	(25)	$\omega_Z<-K/Cn$ (ins.)

and W^* represents the terms of higher degree. We try a function W that does not contain τ explicitly:

$$W=(m_2D_4\lambda^2-m_2D_3\lambda-m_2D_2)\alpha^2+D_1m_2\alpha\beta+m_1\beta^2 \quad (19)$$

where W is positive definite when

$$\lambda<\frac{D_3-\sqrt{D_3^2+4D_3D_4}}{2D_4}$$

or

$$\frac{D_3+\sqrt{D_3^2+4D_3D_4}}{2D_4}<\lambda \quad (20)$$

$$\lambda<\frac{D_3-\sqrt{D_3^2+4D_4[D_2+(D_1^2m_2/4m_1)]}}{2D_4}$$

or

$$\frac{D_3+\sqrt{D_3^2+4D_4[D_2+(D_1^2m_2/4m_1)]}}{2D_4}<\lambda \quad (21)$$

Furthermore,

$$L-W=m_2\Big\{D_4\big[\omega_Z^2(\tau)-\lambda^2\big]+D_3\big[\lambda-\omega_Z(\tau)\big]\Big\}\alpha^2 \quad (22)$$

The condition for $L-W\geq 0$ ($=0$ only when $\alpha=\beta=0$) is

$$\frac{D_3}{D_4}-\omega_Z(\tau)>\lambda>\omega_Z(\tau) \quad \text{or} \quad \omega_Z(\tau)>\lambda>\frac{D_3}{D_4}-\omega_Z(\tau) \quad (23)$$

If Eq. (23) is satisfied, L is positive definite. Since W^* is of higher degree, $L+W^*$ is also positive definite. We choose $m_1\gg m_2$, then Eq. (21) coincides with Eq. (20). Furthermore, from Eqs. (20) and (23), properly choosing λ, we have the range of $\omega_Z(\tau)$:

$$\omega_Z(\tau)<\frac{D_3-\sqrt{D_3^2+4D_3D_4}}{2D_4}$$

or

$$\frac{D_3+\sqrt{D_3^2+4D_3D_4}}{2D_4}<\omega_Z(\tau) \quad (24)$$

The preceding analysis tells us that Eq. (24) is the condition for the positive definiteness of $L+W^*$. When Eq. (24) is satisfied, $\mathrm{d}V/\mathrm{d}\tau$ is positive definite, and V is positive in the region $\alpha>0$ and $\beta>0$. Since V does not contain τ explicitly, it permits an infinitesimal upper limit.[2,3] By Liapunov's instability theorem, the relative equilibrium $\alpha=\beta=0$ is unstable.

If the absolute value of $\omega_Z(\tau)$ is rather small, we may neglect the $\omega_Z^2(\tau)$ term in Eq. (3). Then the differential equations of disturbances become

$$\alpha'=\beta, \quad \beta'=-D_2\alpha-D_3\omega_Z(\tau)\sin\alpha-D_1\beta+\cdots \quad (25)$$

By using the Liapunov functions of the same form as Eqs. (6) and (16) and similar reasoning in Table 1, we can prove that the necessary and sufficient condition of stability of relative equilibrium $\alpha=\beta=0$ is

$$\omega_Z>-K/Cn \quad (26)$$

with $\omega_Z=-K/Cn$ excluded.

Conclusions

The stability of a single-gimbal rate gyro with zero input mounted on a vehicle, which has a time-varying angular velocity component on the reference axis of the spinning of the rotor of the gyro, is studied by Liapunov direct method. Since the differential equation for the disturbances is nonautonomous, the Liapunov functions used must satisfy more conditions than that for an autonomous system. The necessary and sufficient conditions for asymptotic stability are obtained. If the angular momentum of the rotor is sufficiently large, the conditions approximately reduce so that the aforementioned angular velocity component of the vehicle is greater than zero.

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